

# Constraint Propagation of $C^2$ -adjusted Formulation — Another Recipe for Robust ADM Evolution System —

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With a purpose of constructing a robust evolution system against numerical instability for integrating the Einstein equations, we propose a new formulation by adjusting the ADM evolution equations with constraints. We apply an adjusting method proposed by Fiske (2004) which uses the norm of the constraints,  $C^2$ . One of the advantages of this method is that the effective signature of adjusted terms (Lagrange multipliers) for constraint-damping evolution is pre-determined. We demonstrate this fact by showing the eigenvalues of constraint propagation equations. We also perform numerical tests of this adjusted evolution system using polarized Gowdy-wave propagation, which show robust evolutions against the violation of the constraints than that of the standard ADM formulation.

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## I. INTRODUCTION

The standard way for integrating the Einstein equations is to split spacetime into space and time. The Arnowitt-Deser-Misner (ADM) formulation [1, 2] provides the fundamental spacetime decompositions. However, it is known that the set of the ADM evolution equations is not appropriate for numerical simulations such as the coalescences of the binary neutron-stars and/or black-holes, which are the main targets of gravitational wave sources, and which requires quite long-term time integration.

In order to perform an accurate and stable long-term numerical simulation in strong gravitational field, we need to modify the ADM evolution equations. This is called as the “formulation problem in numerical relativity” [3–5].

The origin of the formulation problem is the violation of constraints, which triggers the blow-up of simulations. The discretization of equations arises truncation errors inevitably, so that we have to adjust the evolution system which is robust for error-growing modes. Several formulations are suggested and applied; among them, the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation [6, 7], the generalized-harmonic (GH) formulation [8, 9] and the Kidder-Scheel-Teukolsky (KST) formulation [10] are applied widely for the inspiral black-hole binary mergers. (Many numerical simulations are reported, but we here cite the works [11, 12] for applications of the BSSN formulation, [13] for the GH formulation, and [14]

for the KST formulation). There are also many other formulations which are waiting to be tested [15–19].

The current succeeded large-scale numerical simulations are applying such modern reformulations, but also using the “constraint-damping” technique, which is obtained by adding the constraint terms to evolution equations. The additional constraint-damping terms are reported to be the key implementation in BSSN and GH system (e.g. [20, 21]). We [16, 22, 23] systematically investigated how the additional constraint terms change the original evolution systems, under the name “adjusted systems”. As we will review in Sec.II, monitoring the stability of the evolution is equivalent to check the constraint propagation equations (dynamical equations of constraints). Therefore, we proposed to analyze the eigenvalues of the constraint propagation equations, which can predict the violation of constraints before we try actual simulations.

Based on the same motivation with this “adjusted system”, Fiske [24] proposed an adjustment which uses the norm of constraints,  $C^2$ , which we call the “ $C^2$ -adjusted formulation”. He applied this method to the Maxwell equations, and reported that this method reduces the constraint violations for a certain range of the coefficient. An advantage of this  $C^2$ -adjusted formulation is that the effective signature of the coefficients is pre-determined. In this article, we apply the  $C^2$ -adjusted formulation to the ADM evolution equations, since the ADM formulation is the one of the most basic evolution systems in general relativity. We show the eigenvalue analysis of the constraint propagation of this set, and also demonstrate numerical evolutions.

Before the numerical relativity groups faced the formulation problem, Detweiler [25] suggested another ad-

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justment based on the ADM evolution equations. He proposed a particular combination of adjustments which make the norm of constraints damp down. The story is quite similar to this work. However, Detweiler's method is restricted with the maximal slicing condition,  $K = 0$ , and also the behavior except the flat-space is unknown. We also show numerical demonstrations of Detweiler's evolution equation for a comparison.

We compare the violations of the constraints between the standard ADM, Detweiler's ADM and  $C^2$ -adjusted ADM formulations. We use the polarized Gowdy-wave evolution which is one of the comparison test problems as is known to the Apples-with-Apples testbeds [26]. The models precisely fixed up to the gauge conditions, boundary conditions, and technical parameters, therefore testbeds are often used for comparison between formulations [27–29].

The plan of this article is as follows. We review the idea of adjusted systems and  $C^2$ -adjusted formulation in Sec.II. We also describe a recipe for analyzing the constraint propagation with its eigenvalue analysis which we call the constraint amplification factors (CAFs). In Sec.III, we apply the  $C^2$ -adjusted formulation to the ADM equations and show its CAFs. We also review Detweiler's formulation in Sec.II. We show our numerical evolutions in Sec.IV, and we summarize this article in Sec.V. In this article, we only consider the vacuum space-time, but the inclusion of matter is straightforward.

## II. THE IDEA OF ADJUSTED SYSTEMS AND $C^2$ -ADJUSTED SYSTEMS

### A. The idea of adjusted systems

We review the general procedure of rewriting the evolution equations which we call adjusted systems [15, 16, 22, 23]. Suppose we have dynamical variables  $u^i$  which evolve along with the evolution equations,

$$\partial_t u^i = f(u^i, \partial_j u^i, \dots), \quad (2.1)$$

and suppose also that the system has the (first class) constraint equations,

$$C^a(u^a, \partial_j u^a, \dots) \approx 0. \quad (2.2)$$

We propose to study the properties of the evolution equation of  $C^a$  (which we call the constraint propagation),

$$\partial_t C^a = g(C^a, \partial_i C^a, \dots), \quad (2.3)$$

for predicting the violation behavior of constraints,  $C^a$ , in time evolution. The equation (2.3) is theoretically weakly zero, i.e.  $\partial_t C^a \approx 0$ , since the system is supposed to be the first class. However, the free numerical evolution with the discretized grids introduces constraint violation at least the level of truncation error, which sometimes grows to stop the simulations. The set of the ADM formulation

has such a disastrous feature even in the Schwarzschild spacetime, as was shown in [23].

Such features of the constraint propagation equations, (2.3), will be changed when we modify the original evolution equations. Suppose we add the constraint terms to the right-hand-side of (2.1) as

$$\partial_t u^i = f(u^i, \partial_j u^i, \dots) + F(C^a, \partial_j C^a, \dots), \quad (2.4)$$

where  $F(C^a, \dots) \approx 0$  in principle but not exactly zero in numerical evolutions, then (2.3) will also be modified as

$$\partial_t C^a = g(C^a, \partial_i C^a, \dots) + G(C^a, \partial_i C^a, \dots). \quad (2.5)$$

Therefore we are able to control  $\partial_t C^a$  by an appropriate adjustment  $F(C^a, \partial_j C^a, \dots)$  in (2.4). There exist various combinations of  $F(C^a, \partial_j C^a, \dots)$  in (2.4), and all the alternative formulations are using this technique. Therefore, our goal is to find out a better way of adjusting the evolution equations which realizes  $\partial_t C^a \leq 0$ .

### B. The idea of $C^2$ -adjusted formulations

Fiske [24] proposed an adjustment of the evolution equations in the way of

$$\partial_t u^i = f(u^i, \partial_j u^i, \dots) - \kappa^{ij} \left( \frac{\delta C^2}{\delta u^j} \right), \quad (2.6)$$

where  $\kappa^{ij}$  is positive-definite constant coefficient, and  $C^2$  is the norm of constraints which is defined as  $C^2 \equiv \int C_a C^a d^3x$ . The term  $(\delta C^2 / \delta u^j)$  is the functional derivative of  $C^2$  with  $u^j$ . We call the set of (2.6) with (2.2) as “ $C^2$ -adjusted formulation”. The associated constraint propagation equation becomes

$$\partial_t C^2 = h(C^a, \partial_i C^a, \dots) - \int d^3x \left( \frac{\delta C^2}{\delta u^i} \right) \kappa^{ij} \left( \frac{\delta C^2}{\delta u^j} \right). \quad (2.7)$$

If we set  $\kappa^{ij}$  so as the second term in the RHS of (2.7) becomes dominant than the first term, then  $\partial_t C^2$  becomes negative, which indicates that constraint violations are expected to decay to zero. Fiske presented some numerical examples in the Maxwell system, and concluded that this method actually reduces the constraint violations. He also reported that the coefficient  $\kappa^{ij}$  has a practical upper limit in order not to crash simulations.

### C. The idea of CAFs

There are many efforts of re-formulation of the Einstein equations which make the evolution equations in an explicit first-order hyperbolic form (e.g. [10, 17, 30, 31]). This is motivated by the expectations that the symmetric

hyperbolic system has well-posed properties in its Cauchy treatment in many systems and that the boundary treatment can be improved if we know the characteristic speed of the system. The advantage of the standard ADM system [2] (compared with the original ADM system [1]) is reported by Frittelli [32] from the point of the hyperbolicity of the constraint propagation equations. However, the classification of hyperbolicity (weakly, strongly or symmetric hyperbolic) only uses the characteristic part of evolution equations and ignore the rest. Several numerical experiments [3, 33] reported that such a classification is not enough to predict the stability of the evolution system, especially for highly non-linear system like the Einstein equations.

In order to investigate the stability structure of (2.5), the authors [22] proposed the constraint amplification factors (CAFs). The CAFs are the eigenvalues of the coefficient matrix,  $M^a_b$  (below), which is the Fourier-transformed components of the constraint propagation equations,  $\partial_t \hat{C}^a$ . That is,

$$\partial_t \hat{C}^a = g(\hat{C}^a) = M^a_b \hat{C}^b, \quad \text{where } C^a(x, t) = \int \hat{C}(k, t)^a \exp(ik \cdot x) d^3k. \quad (2.8)$$

CAFs include all the contributions of the terms, and enable us to check the eigenvalues. If CAFs have negative real-part, the constraints are forced to be diminished. Therefore, we expect more stable evolution than a system which has CAFs with positive real-part. If CAFs have non-zero imaginary-part, the constraints are supposed to propagate away. Therefore, we expect more stable evolution than a system which has CAFs with zero imaginary-part. The discussion and examples are shown in [3, 15], where several adjusted-ADM systems [3] and adjusted-BSSN systems [16] are proposed.

### III. APPLICATION TO THE ADM FORMULATION

#### A. The standard ADM formulation and $C^2$ -adjusted ADM formulation

We start by presenting the standard ADM formulation [2] of the Einstein equations. The standard ADM evolution equations are written as

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i, \quad (3.1)$$

$$\partial_t K_{ij} = \alpha({}^{(3)}R_{ij} + K K_{ij} - 2K_{i\ell} K^\ell_j) - D_i D_j \alpha + K_{\ell i} D_j \beta^\ell + K_{\ell j} D_i \beta^\ell + \beta^\ell D_\ell K_{ij}, \quad (3.2)$$

where  $(\gamma_{ij}, K_{ij})$  are the induced three-metric and the extrinsic curvature,  $(\alpha, \beta^i)$  are the lapse function and the shift vector,  $D_i$  is the covariant derivative associated with  $\gamma_{ij}$  and  ${}^{(3)}R_{ij}$  is the three Ricci tensor. The constraint

equations are

$$\mathcal{H} \equiv {}^{(3)}R + K^2 - K_{ij} K^{ij} \approx 0, \quad (3.3)$$

$$\mathcal{M}_i \equiv D_j K^j_i - D_i K \approx 0, \quad (3.4)$$

where  ${}^{(3)}R$  is the three-scalar curvature,  ${}^{(3)}R = \gamma^{ij} {}^{(3)}R_{ij}$  and  $K$  is the trace-part of the extrinsic curvature,  $K = \gamma^{ij} K_{ij}$ .

The constraint propagation equations of the Hamiltonian constraint, (3.3), and the momentum constraints, (3.4), can be written as

$$\partial_t \mathcal{H} = \beta^i D_i \mathcal{H} - 2\alpha D_i \mathcal{M}^i + 2\alpha K \mathcal{H} - 4(D_i \alpha) \mathcal{M}^i, \quad (3.5)$$

$$\partial_t \mathcal{M}_i = -(1/2)\alpha D_i \mathcal{H} + \beta^\ell D_\ell \mathcal{M}_i - (D_i \alpha) \mathcal{H} + (D_i \beta^\ell) \mathcal{M}_\ell + \alpha K \mathcal{M}_i, \quad (3.6)$$

respectively.

Now we apply  $C^2$ -adjustment to the ADM formulation, which can be written as

$$\partial_t \gamma_{ij} = (3.1) - \kappa_{\gamma ijmn} \left( \frac{\delta C^2}{\delta \gamma_{mn}} \right), \quad (3.7)$$

$$\partial_t K_{ij} = (3.2) - \kappa_{K ijmn} \left( \frac{\delta C^2}{\delta K_{mn}} \right), \quad (3.8)$$

where  $C^2$  is the norm of the constraints, which we set

$$C^2 \equiv \int (\mathcal{H}^2 + \gamma^{ij} \mathcal{M}_i \mathcal{M}_j) d^3x, \quad (3.9)$$

and both coefficients of  $\kappa_{\gamma ijmn}, \kappa_{K ijmn}$  are supposed to be positive definite. We write  $(\delta C^2 / \delta \gamma_{mn})$  and  $(\delta C^2 / \delta K_{mn})$  explicitly as (A1) and (A2) in Appendix A.

#### B. Constraint Propagation with $C^2$ -adjusted ADM formulation

In this subsection, we discuss the constraint propagation of the  $C^2$ -adjusted ADM formulation, by giving the CAFs on flat background metric. We show CAFs are negative real numbers or complex numbers with negative real-part.

The constraint propagation equations, (3.5) and (3.6), are changed due to  $C^2$ -adjusted terms. The full expressions of the constraint propagation equations are shown as (B1) and (B11) in Appendix B.

If we fix the background is flat spacetime,  $(\alpha = 1, \beta^i = 0, \gamma_{ij} = \delta_{ij}, K_{ij} = 0)$ , then CAFs are easily derived. For simplicity, we also set  $\kappa_{\gamma ijmn} = \kappa_{K ijmn} = \kappa \delta_{im} \delta_{jn}$ , where  $\kappa$  is positive. The Fourier-transformed equations of the constraint propagation equations are

$$\partial_t \begin{pmatrix} \hat{\mathcal{H}} \\ \hat{\mathcal{M}}_i \end{pmatrix} = \begin{pmatrix} -4\kappa |\vec{k}|^4 & -2ik_j \\ -(1/2)ik_i & \kappa(-|\vec{k}|^2 \delta_{ij} - 3k_i k_j) \end{pmatrix} \begin{pmatrix} \hat{\mathcal{H}} \\ \hat{\mathcal{M}}_j \end{pmatrix}. \quad (3.10)$$

The eigenvalues,  $\lambda$ , of the coefficient matrix of (3.10) are given by solving

$$(\lambda + \kappa|k|^2)^2(\lambda^2 + A\lambda + B) = 0,$$

where  $A \equiv 4\kappa|k|^2(|k|^2 + 1)$  and  $B \equiv |k|^2 + 16\kappa^2|k|^6$ . Therefore, the four eigenvalues are

$$(-\kappa|k|^2, -\kappa|k|^2, \lambda_+, \lambda_-), \quad (3.11)$$

where

$$\lambda_{\pm} = -2\kappa|k|^2(|k|^2 + 1) \pm |k|\sqrt{-1 + 4\kappa^2|k|^2(|k|^2 - 1)^2}. \quad (3.12)$$

From the relation of the coefficients with solutions,

$$\lambda_+ + \lambda_- = -A < 0, \quad \text{and} \quad \lambda_+ \lambda_- = B > 0, \quad (3.13)$$

we find both the real parts of  $\lambda_+$  and  $\lambda_-$  are negative. Therefore, we see all four eigenvalues are complex numbers with negative real-part or negative real numbers.

On the other hand, the CAFs of the standard ADM formulation on flat background [ $\kappa = 0$  in (3.11)] are reduced to

$$(0, 0, \pm i|\vec{k}|), \quad (3.14)$$

where the real-part of all of the CAFs are zero. Therefore the introduction of the  $C^2$ -adjusted terms to the evolution equations changes the constraint propagation equations to a self-decay system.

More precisely, CAFs depend to  $|k|^2$  if  $\kappa \neq 0$ . This indicates that adjusted terms affect to reduce high frequency error-growing modes. Since we intend not to change the original evolution equations drastically by adding adjusted terms, we consider only small  $\kappa$ . This limits the robustness of the system to the low frequency error-growing modes. Therefore the system may stop due to the low frequency modes, but the longer evolutions are expected to be obtained.

### C. Detweiler's ADM formulation

We review Detweiler's ADM formulation [25] for a comparison with the  $C^2$ -adjusted ADM formulation and the standard ADM formulation. Detweiler proposed an evolution system in order to ensure the decay of the norm of constraints,  $\partial_t C^2 < 0$ . His system can be treated as one of the adjusted ADM systems and the set of evolution equations can be written as

$$\partial_t \gamma_{ij} = (3.1) + LD_{\gamma_{ij}}, \quad (3.15)$$

$$\partial_t K_{ij} = (3.2) + LD_{K_{ij}}, \quad (3.16)$$

$$\text{where} \quad D_{\gamma_{ij}} \equiv -\alpha^3 \gamma_{ij} \mathcal{H}, \quad (3.17)$$

$$\begin{aligned} D_{K_{ij}} \equiv & \alpha^3 (K_{ij} - (1/3)K\gamma_{ij})\mathcal{H} \\ & + \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_\ell \alpha)\gamma_{ij}\gamma^{k\ell}]\mathcal{M}_k \\ & + \alpha^3 [\delta^k_{(i}\delta^{\ell}_{j)} - (1/3)\gamma_{ij}\gamma^{k\ell}]D_k \mathcal{M}_\ell, \end{aligned} \quad (3.18)$$

where  $L$  is a constant. He found that with this particular combination of adjustments, the evolution of the norm constraints,  $C^2$ , can be negative definite when we apply the maximal slicing condition,  $K = 0$ , for fixing the lapse function,  $\alpha$ . Note that the effectiveness with other gauge conditions is remain unknown. The numerical demonstrations with Detweiler's ADM formulation are presented in [5, 22], and there we can see the drastic improvements for stability.

The CAFs of Detweiler's ADM formulation on flat background metric are derived as [22],

$$\begin{aligned} & \left( -(L/2)|\vec{k}|^2, -(L/2)|\vec{k}|^2, \right. \\ & \left. - (4L/3)|\vec{k}|^2 \pm \sqrt{|\vec{k}|^2\{-1 + (4/9)L^2|\vec{k}|^2\}} \right), \end{aligned} \quad (3.19)$$

which indicates the constraints will damp down if  $L > 0$ , apparently better feature than the standard ADM formulation.

## IV. NUMERICAL EXAMPLES

We demonstrate the damping of constraint violations in numerical evolutions using the polarized Gowdy-wave spacetime, which is one of the standard tests for comparisons of formulations in numerical relativity as is known as the Apples-with-Apples testbeds[26]. The tests have been used by several groups and were reported in the same manner (e.g. [27–29]).

The testbeds provide three tests of the solutions of the Einstein equations: gauge-wave, linear-wave, and Gowdy-wave tests. Among these tests, we report only on the Gowdy-wave test. This is because the other two are based on the flat backgrounds and the violations of constraints are already small, so that the differences of evolutions between the ADM,  $C^2$ -adjusted ADM, and Detweiler-ADM are indistinguishable.

### A. Gowdy-wave Testbed

The metric of the polarized Gowdy-wave is given by

$$ds^2 = t^{-1/2}e^{\lambda/2}(-dt^2 + dx^2) + t(e^P dy^2 + e^{-P} dz^2), \quad (4.1)$$

where  $P$  and  $\lambda$  are functions of  $x$  and  $t$ . The time coordinate  $t$  is chosen such that time increases as the universe expands, this metric is singular at  $t = 0$  which corresponds to the cosmological singularity.

For simple forms of the solutions,  $P$  and  $\lambda$ , are given by

$$P = J_0(2\pi t) \cos(2\pi x), \quad (4.2)$$

$$\begin{aligned} \lambda = & -2\pi t J_0(2\pi t) J_1(2\pi t) \cos^2(2\pi x) + 2\pi^2 t^2 [J_0^2(2\pi t) \\ & + J_1^2(2\pi t)] - (1/2)\{(2\pi)^2[J_0^2(2\pi) + J_1^2(2\pi)] \\ & - 2\pi J_0(2\pi) J_1(2\pi)\}, \end{aligned} \quad (4.3)$$

where  $J_n$  is the Bessel function.

Following [26], the new time coordinate  $\tau$ , which satisfies the harmonic slicing, is obtained by coordinate transformation as

$$t(\tau) = ke^{c\tau}, \quad (4.4)$$

where  $k$  and  $c$  are arbitrary constants. We also follow [26] for choosing these constants  $k$ ,  $c$  and initial time  $t_0$  as

$$k \sim 9.67076981276405, \quad c \sim 0.002119511921460, \quad (4.5)$$

$$t_0 = 9.87532058290982 \quad (4.6)$$

in such a way that the lapse function in new time coordinate is unity and  $t = \tau$  at initial time.

We also use following parameters specified in [26],

- Simulation domain:  $x \in [-0.5, 0.5], y = z = 0$ .
- Grid:  $x_n = -0.5 + (n - (1/2))dx$ ,  $n = 1, \dots, 100$ , where  $dx = 1/100$ .
- Time step:  $dt = 0.25dx$ .
- Boundary conditions: Periodic boundary condition in  $x$ -direction and planar symmetry in  $y$ - and  $z$ -directions.
- Gauge conditions: the harmonic slicing and  $\beta^i = 0$ .
- Scheme: second order iterative Crank-Nicholson.

Our code passed convergence tests with the second-order accuracy.

### B. Constraint violations and the damping of the violations

Figure 1 shows the L2 norm of the Hamiltonian constraint and momentum constraints with a function of backward time ( $-t$ ) in the case of the standard ADM formulation, (3.1)-(3.2). We see the violations of the momentum constraints are larger than that of the Hamiltonian constraint at the initial stage, and both grow larger with time. The behavior is well-known, and the starting point of the formulation problem.

We, then, compare the evolutions with three formulations: (a) the standard ADM formulation (3.1)-(3.2), (b) Detweiler's formulation (3.15)-(3.16), and (c) the  $C^2$ -adjusted ADM formulation (3.7)-(3.8). We tuned the parameters  $L$  in (a), and  $\kappa_{\gamma ijmn}$  and  $\kappa_{K ijmn}$  in (c) within the expected ranges from the eigenvalue analyses. In the formulation (c), we set  $\kappa_{\gamma ijmn} = \kappa_{\gamma} \delta_{im} \delta_{jn}$  and  $\kappa_{K ijmn} = \kappa_K \delta_{im} \delta_{jn}$  for simplicity, and optimized  $\kappa_{\gamma}$  and  $\kappa_K$  in their positive ranges. We use  $L = -10^{+1.9}$  and  $(\kappa_{\gamma}, \kappa_K) = (-10^{-9.0}, -10^{-3.5})$  for the plots, since the violation of constraints are minimized at  $t = -1000$  for those evolutions. Note that the signatures of  $(\kappa_{\gamma}, \kappa_K)$

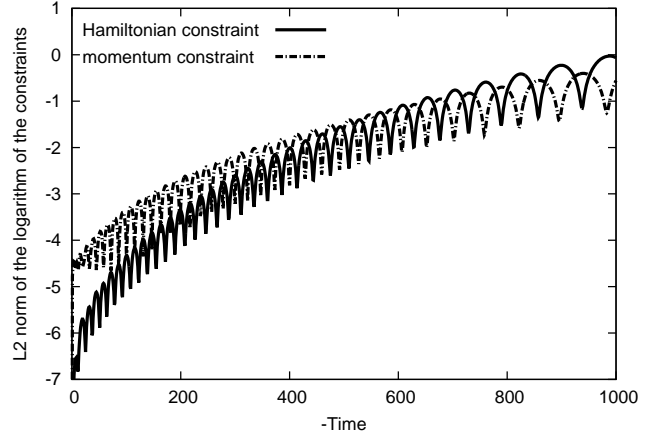


FIG. 1. The L2 norm of the Hamiltonian and momentum constraints of the Gowdy-wave evolution using the standard ADM formulation. We see that the violation of the momentum constraints is larger initially, and both violations are growing with time.

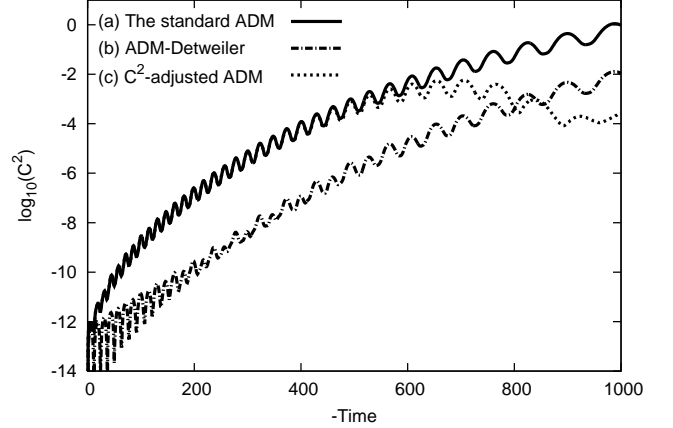


FIG. 2. The L2 norm of the constraints,  $C^2$ , of the polarized Gowdy-wave tests with ADM and two types of adjusted formulations. The vertical axis is the logarithm of the  $C^2$  and the horizontal axis is backward time. The solid line (a) is of the standard ADM formulation. The dot-dashed line (b) is the evolution with Detweiler's ADM with  $L = -10^{+1.9}$ . The dotted line (c) is the  $C^2$ -adjusted ADM with  $\kappa_{\gamma} = -10^{-9.0}$  and  $\kappa_K = -10^{-3.5}$ . We see the lines (a) and (c) almost overlap until  $t = -500$ , then the case (c) keeps the L2 norm at the level  $\leq 10^{-3}$ , while the lines of (a) and (b) monotonically grow larger with oscillations. We confirmed this behavior up to  $t \simeq -1700$ .

and  $L$  are reversed from the expected one in Sec. II and Sec. III, respectively, since we integrate time backward.

We plot the L2 norms of  $C^2$  of these three formulations in Figure 2. We see the constraint violations of (a)(the standard ADM formulation) and (b)(Detweiler's formulation) grow larger with time, while that of (c)( $C^2$ -adjusted ADM formulation) almost coincide with (a) until  $t = -500$ , then the violation of (c) begins smaller than



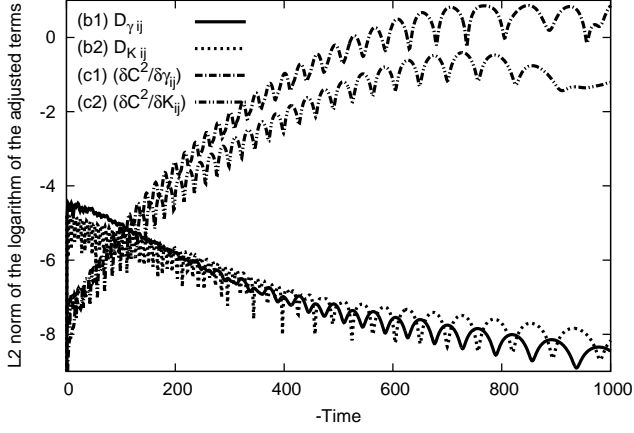


FIG. 3. The magnitudes of the adjusted terms in each equations for the evolutions shown in Figure 2. The vertical axis is the logarithm of the adjusted terms. The horizontal axis is backward time. The lines (b1) and (b2) are the adjusted terms (3.17) and (3.18) respectively. The lines (c1) and (c2) are the adjusted terms (A1) and (A2) respectively. We see the adjustments in Detweiler-ADM [the lines (b1) and (b2)] decrease with time, which indicates that these contributions become less effective.

(a). The L2 violation level of (c), then, keeps its magnitude at most  $O(10^{-3})$ , while those of (a) and (b) monotonically grow larger with oscillations. Figure 2 shows up to  $t = -1000$ , but we confirmed this behavior up to  $t = -1700$ .

Figure 2 tells us that the effects of Detweiler’s adjustment appear at the initial stage, while  $C^2$ -adjustment contributes at the later stage. The time difference can be seen also from the magnitudes of adjustment terms in each evolution equations, which we show in Figure 3. The lines (b1), (b2), (c1), and (c2) are the norms of  $D_{\gamma_{ij}}$  in (3.17),  $D_{K_{ij}}$  in (3.18),  $\delta C^2 / \delta \gamma_{ij}$  in (A1), and  $\delta C^2 / \delta K_{ij}$  in (A2), respectively.

We see that the L2 norms of the adjusted terms of Detweiler’s ADM formulation,  $D_{\gamma_{ij}}$  and  $D_{K_{ij}}$ , decrease, while that of the  $C^2$ -adjusted ADM formulation increase. If the magnitudes of the adjusted terms are smaller, the effects of the constraint damping become small. Therefore, the L2 norm of  $C^2$  of Detweiler’s ADM formulation are not damped down in the later stage in Figure 2.

One possible explanation for the weak effect of Detweiler’s adjustment in the later stage is the existence of the lapse function,  $\alpha$  (and  $\alpha^2$ ,  $\alpha^3$ ), in the adjusted terms in (3.17)–(3.18). The Gowdy-wave testbed is the evolution to the initial singularity of the space-time, and the lapse function becomes smaller with evolution. Note that in previous works [5, 22], we see that the constraint violations are damped down in the simulation with Detweiler’s ADM formulation, where the lapse function,  $\alpha$ , is adopted by the geodesic condition.

In Figure 4, we plotted the magnitude of the original terms and the adjusted terms of  $C^2$ -adjusted ADM for-

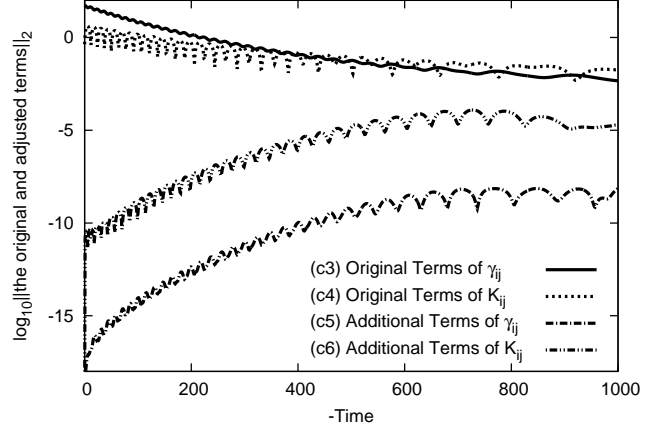


FIG. 4. Comparison of the magnitude of the original terms and the adjusted terms of the  $C^2$ -adjusted ADM formulation, (3.7)–(3.8). The lines (c3) and (c4) are the L2 norm of the original terms [the evolution equations of  $g_{ij}$  and  $K_{ij}$ , (3.1) and (3.2)], respectively. The lines (c5) and (c6) are the L2 norm of the adjusted terms, which is the second terms of the right-hand side of (3.7) and (3.8), respectively. We see the adjusted terms are “tiny”, compared with the original terms.

mulation; the first and second terms in (3.7) and (3.8). We find that there is  $O(10^2)$ – $O(10^5)$  of differences between them. Therefore, we conclude that the adjustments do not disturb the original ADM formulation, but control the violation of the constraints. We may understand that higher derivative terms in (A1) and (A2) work as artificial viscosity terms in numerics.

### C. Parameter dependence of the $C^2$ -adjusted ADM formulation

There are two parameters,  $\kappa_\gamma$  and  $\kappa_K$ , in the  $C^2$ -adjusted ADM formulation and we next study the sensitivity of these two on the damping effect to the constraint violation.

Figure 5 shows the dependences on  $\kappa_\gamma$  and  $\kappa_K$ . In Figure 5 (A), we fix  $\kappa_K = 0$  and change  $\kappa_\gamma$ . In Figure 5 (B), we fix  $\kappa_\gamma = 0$  and change  $\kappa_K$ . In Figure 5 (A), we see that all the simulations stop soon after the damping effect appears. On the other hand, in Figure 5 (B), we see that the simulations continue with constraint-damping effects. These results suggest  $\kappa_K \neq 0$  or  $\kappa_\gamma = 0$  is essential to keep the constraint-damping effects.

We think the trigger for stopping evolutions in the cases of Figure 5 (A) (when  $\kappa_K = 0$ ) is the term  $\mathbb{H}_5^{abcd}(\partial_a \partial_b \partial_c \partial_d \mathcal{H})$  which appears in the constraint propagation equation of the Hamiltonian constraint, (B1). We evaluated and checked each terms and found that  $\mathbb{H}_5^{abcd}$  exponentially grows in time and dominates the other terms in (B1) before the simulation stops. Since  $\mathbb{H}_5^{abcd}$  consists of  $\gamma^{ij}\gamma^{mn}$  [see (A5) and (B6)], the

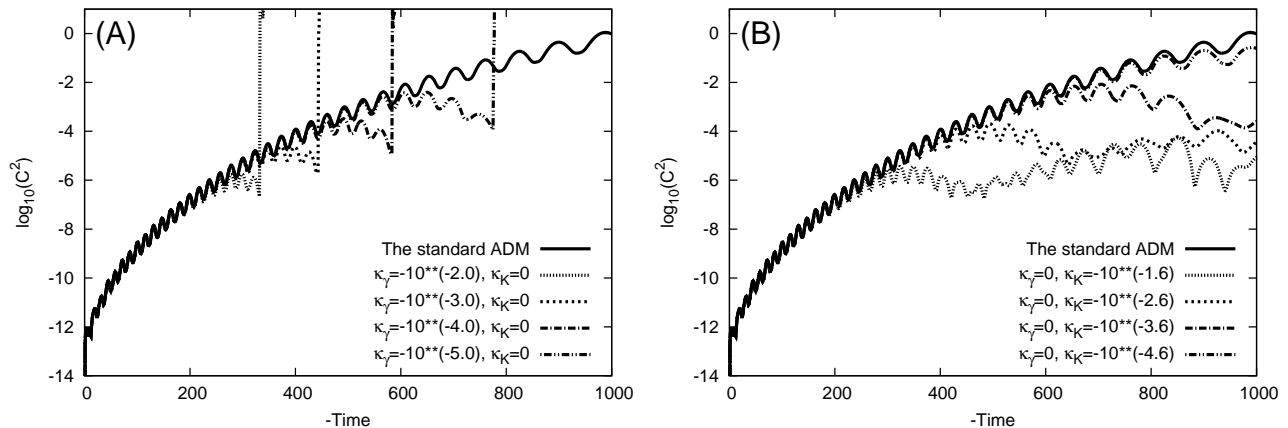


FIG. 5. Parameter dependence of the  $C^2$ -adjusted ADM formulation. The vertical axis is the logarithm of the  $C^2$  and the horizontal axis is backward time. The left panel (A) is the evolutions with  $\kappa_K = 0$  and  $\kappa_\gamma = -10^{-2.0}, -10^{-3.0}, -10^{-4.0}, -10^{-5.0}$ . The right panel (B) is the cases with  $\kappa_\gamma = 0$  and  $\kappa_K = -10^{-1.6}, -10^{-2.6}, -10^{-3.6}, -10^{-4.6}$ . In (A), we see that the simulations stop soon after the constraint dumping effect appears. In (B), we see that the simulations continue with constraint-damping effects.

time backward integration of Gowdy spacetime makes this term disastrous. So that, in this Gowdy testbed, the cases  $\kappa_\gamma = 0$  reduce this trouble and keep the evolution with constraint-damping effects.

The sudden stops of evolutions in Figure 5 (A) can be interpreted due to a non-linear growth of “constraint shocks”, since the adjusted terms are highly non-linear<sup>1</sup>. The robustness against a constraint-shock is hard to be proved, but the continuous evolution cases in Figure 5 (B) may show that a remedial example is available by tuning parameters.

## V. SUMMARY

In order to construct a robust and stable formulation, we proposed a new set of evolution equations, which we call the  $C^2$ -adjusted ADM formulation. We applied the adjusting method suggested by Fiske [24] to the ADM formulation. We obtained the evolution equations as (A1)-(A2) and the constraint propagation equations, (B1) and (B11), and also discussed the constraint propagation of this system. We analyzed the constraint amplification factors (CAFs) on the flat background, and confirmed that all of the CAFs have negative real-part which indicate the damping of the constraint violations. We, then, performed numerical tests with the polarized Gowdy-wave and showed the damping of the constraint violations as expected.

There are two advantages of the  $C^2$ -adjusted system. One is that we can uniquely determine the form of the adjustments. The other is that we can specify the effective signature of the coefficients (Lagrange multipliers)

independent on the background. (The term “effective” means that the system has the property of the damping constraint violations). In the previous our study [22], we systematically examined several combinations of adjustments to the ADM evolution equations, and discuss the effective signature of those Lagrange multipliers using CAFs as the guiding principle. However, the  $C^2$ -adjusted idea, (2.6), automatically includes this guiding principle. We confirm this fact using CAF-analysis on the flat background.

The  $C^2$ -adjusted idea is the one of the useful ideas to decide the adjustments with theoretical logic. We are now applying this idea also to the BSSN formulation which will be presented elsewhere near future.

We performed the simulation with the  $C^2$ -adjusted ADM formulation on the Gowdy-wave spacetime and confirmed the effect of the constraint dumping. We investigated the parameter dependencies and found that the constraint-damping effect does not continue due to one of the adjusted terms. We also found that the Detweiler’s adjustment [25] is not so effective against constraint violations on this spacetime. Up to this moment, we do not yet know how to choose the ranges of parameters which are suitable to damp the constraint violations unless the simulations are actually performed.

It would be helpful if there are methods to monitor the order of constraint violations and to maintain them by tuning the Lagrange multipliers automatically. Such an implementation would make numerical relativity more friendly to the beginners. Applications of the controlling theories in this direction are in progress.

<sup>1</sup> We appreciate the anonymous referee for pointing out this issue.

## ACKNOWLEDGMENTS

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## Appendix A: The additional $C^2$ -adjusted terms

The adjusted terms  $\delta C^2/\delta\gamma_{mn}$  and  $\delta C^2/\delta K_{mn}$  in (3.7)-(3.8) are written as

$$\begin{aligned} \frac{\delta C^2}{\delta\gamma_{mn}} &= 2H_1^{mn}\mathcal{H} - 2(\partial_\ell H_2^{mn\ell})\mathcal{H} - 2H_2^{mn\ell}(\partial_\ell\mathcal{H}) + 2(\partial_k\partial_\ell H_3^{mnk\ell})\mathcal{H} + 4(\partial_\ell H_3^{mnk\ell})(\partial_k\mathcal{H}) + 2H_3^{mnk\ell}(\partial_k\partial_\ell\mathcal{H}) \\ &\quad + 2M_{1i}^{mn}\mathcal{M}^i - 2(\partial_\ell M_{2i}^{mn\ell})\mathcal{M}^i - 2M_{2i}^{mn\ell}(\partial_\ell\mathcal{M}^i) - \mathcal{M}^m\mathcal{M}^n, \end{aligned} \quad (\text{A1})$$

$$\frac{\delta C^2}{\delta K_{mn}} = 2H_4^{mn}\mathcal{H} + 2M_{3i}^{mn}\mathcal{M}^i - 2(\partial_\ell M_{4i}^{mn\ell})\mathcal{M}^i - 2M_{4i}^{mn\ell}(\partial_\ell\mathcal{M}^i), \quad (\text{A2})$$

where

$$\begin{aligned} H_1^{mn} &= -2R^{mn} + {}^{(3)}\Gamma^m({}^{(3)}\Gamma^n - {}^{(3)}\Gamma^{meb}{}^{(3)}\Gamma_{eb}^n \\ &\quad - 2KK^{mn} + 2K^m{}_j K^{nj}, \end{aligned} \quad (\text{A3})$$

$$\begin{aligned} H_2^{imn} &= -\gamma^{\ell m}({}^{(3)}\Gamma^n - \gamma^{\ell n}({}^{(3)}\Gamma^m + \gamma^{mn}({}^{(3)}\Gamma^\ell + {}^{(3)}\Gamma^{nm\ell} \\ &\quad + {}^{(3)}\Gamma^{mn\ell} - {}^{(3)}\Gamma^{\ell nm}), \end{aligned} \quad (\text{A4})$$

$$H_3^{ijmn} = \frac{1}{2}\gamma^{m\ell}\gamma^{nk} + \frac{1}{2}\gamma^{km}\gamma^{n\ell} - \gamma^{k\ell}\gamma^{mn}, \quad (\text{A5})$$

$$H_4^{mn} = 2\gamma^{mn}K - 2K^{mn}, \quad (\text{A6})$$

$$\begin{aligned} M_1^{mn}{}_i &= -\frac{1}{2}K_{\ell i,j}\gamma^{jm}\gamma^{\ell n} - \frac{1}{2}K_{\ell i,j}\gamma^{jn}\gamma^{\ell m} + \frac{1}{2}({}^{(3)}\Gamma^n K^m{}_i \\ &\quad + \frac{1}{2}({}^{(3)}\Gamma^m K^n{}_i + {}^{(3)}\Gamma^{amn}K_{ai} - \frac{1}{2}K^{mc}\gamma^{nb}\gamma_{bc,i} \\ &\quad - \frac{1}{2}K^{nc}\gamma^{mb}\gamma_{bc,i} + K_{ab,i}\gamma^{am}\gamma^{bn}), \end{aligned} \quad (\text{A7})$$

$$\begin{aligned} M_2^{\ell mn}{}_i &= -\frac{1}{2}\gamma^{n\ell}K^m{}_i - \frac{1}{2}\gamma^{m\ell}K^n{}_i + \frac{1}{2}\gamma^{mn}K^\ell{}_i \\ &\quad + \frac{1}{2}K^{nm}\delta^\ell{}_i, \end{aligned} \quad (\text{A8})$$

$$M_3^{mn}{}_i = -\frac{1}{2}({}^{(3)}\Gamma^m\delta^n{}_i - \frac{1}{2}({}^{(3)}\Gamma^n\delta^m{}_i + \frac{1}{2}\gamma^{na}\gamma^{mb}\gamma_{ab,i}), \quad (\text{A9})$$

$$M_4^{\ell mn}{}_i = \frac{1}{2}\gamma^{\ell m}\delta^n{}_i + \frac{1}{2}\gamma^{\ell n}\delta^m{}_i - \gamma^{mn}\delta^\ell{}_i, \quad (\text{A10})$$

$H_1^{mn}$ ,  $H_2^{imn}$ ,  $H_3^{ijmn}$ ,  $H_4^{mn}$ ,  $M_1^{mn}{}_i$ ,  $M_2^{jmn}{}_i$ ,  $M_3^{mn}{}_i$ ,  $M_4^{jmn}{}_i$  are the same with the appendix of [23] if  $(m, n) = (n, m)$ .

## Appendix B: The constraint propagation equations of $C^2$ -adjusted ADM formulation

The propagation equation of the Hamiltonian constraint with  $C^2$ -adjusted ADM formulation can be written as

$$\begin{aligned} \partial_t\mathcal{H} &= \mathbb{H}_1\mathcal{H} + \mathbb{H}_2{}^a(\partial_a\mathcal{H}) + \mathbb{H}_3{}^{ab}(\partial_a\partial_b\mathcal{H}) + \mathbb{H}_4{}^{abc}(\partial_a\partial_b\partial_c\mathcal{H}) + \mathbb{H}_5{}^{abcd}(\partial_a\partial_b\partial_c\partial_d\mathcal{H}) + \mathbb{H}_{6a}\mathcal{M}^a + \mathbb{H}_{7a}{}^b(\partial_b\mathcal{M}^a) \\ &\quad + \mathbb{H}_{8a}{}^{bc}(\partial_b\partial_c\mathcal{M}^a) + \mathbb{H}_{9a}{}^{bcd}(\partial_b\partial_c\partial_d\mathcal{M}^a), \end{aligned} \quad (\text{B1})$$



where

$$\begin{aligned}
\mathbb{H}_1 = & 2\alpha K - 2\kappa_{\gamma mnij} \left\{ H_1^{mn} H_1^{ij} - H_1^{mn} (\partial_c H_2^{ijc}) + H_1^{mn} (\partial_d \partial_c H_3^{ijdc}) + H_2^{mn\ell} (\partial_\ell H_1^{ij}) - H_2^{mn\ell} (\partial_\ell \partial_c H_2^{ijc}) \right. \\
& + H_2^{mn\ell} (\partial_\ell \partial_d \partial_c H_3^{ijdc}) + H_3^{mnk\ell} (\partial_k \partial_\ell H_1^{ij}) - H_3^{mnk\ell} (\partial_k \partial_\ell \partial_c H_2^{ijc}) + H_3^{mnk\ell} (\partial_k \partial_\ell \partial_d \partial_c H_3^{ijdc}) \left. \right\} \\
& - 2(\partial_\ell \kappa_{\gamma mnij}) \left\{ H_2^{mn\ell} H_1^{ij} - H_2^{mn\ell} (\partial_c H_2^{ijc}) + H_2^{mn\ell} (\partial_d \partial_c H_3^{ijdc}) + 2H_3^{mn\ell k} (\partial_k H_1^{ij}) \right. \\
& - 2H_3^{mn\ell k} (\partial_k \partial_c H_2^{ijc}) + 2H_3^{mn\ell k} (\partial_k \partial_d \partial_c H_3^{ijdc}) \left. \right\} - 2(\partial_k \partial_\ell \kappa_{\gamma mnij}) \left\{ H_3^{mnk\ell} H_1^{ij} - H_3^{mnk\ell} (\partial_c H_2^{ijc}) \right. \\
& + H_3^{mnk\ell} (\partial_d \partial_c H_3^{ijdc}) \left. \right\} - 2\kappa_{Kmnij} H_4^{mn} H_4^{ij}, \tag{B2}
\end{aligned}$$

$$\begin{aligned}
\mathbb{H}_2^a = & \beta^a - 2\kappa_{\gamma mnij} \left\{ -H_1^{mn} H_2^{ija} + 2H_1^{mn} (\partial_c H_3^{ijac}) + H_2^{mna} H_1^{ij} - H_2^{mna} (\partial_c H_2^{ijc}) - H_2^{mn\ell} (\partial_\ell H_2^{ija}) \right. \\
& + H_2^{mna} (\partial_d \partial_c H_3^{ijdc}) + 2H_2^{mn\ell} (\partial_\ell \partial_c H_3^{ijac}) + H_3^{mnal} (\partial_\ell H_1^{ij}) + H_3^{mnka} (\partial_k H_1^{ij}) - H_3^{mnal} (\partial_\ell \partial_c H_2^{ijc}) \\
& - H_3^{mnka} (\partial_k \partial_c H_2^{ijc}) - H_3^{mnk\ell} (\partial_k \partial_\ell H_2^{ija}) + H_3^{mnal} (\partial_\ell \partial_d \partial_c H_3^{ijdc}) + H_3^{mnka} (\partial_k \partial_d \partial_c H_3^{ijdc}) \\
& + 2H_3^{mnk\ell} (\partial_k \partial_\ell \partial_c H_3^{ijac}) \left. \right\} - 2(\partial_\ell \kappa_{\gamma mnij}) \left\{ -H_2^{mn\ell} H_2^{ija} + 2H_2^{mn\ell} (\partial_c H_3^{ijac}) + 2H_3^{mn\ell a} H_1^{ij} \right. \\
& - 2H_3^{mn\ell a} (\partial_c H_2^{ijc}) - 2H_3^{mn\ell k} (\partial_k H_2^{ija}) + 2H_3^{mn\ell a} (\partial_d \partial_c H_3^{ijdc}) + 4H_3^{mn\ell k} (\partial_k \partial_c H_3^{ijac}) \left. \right\} \\
& - 2(\partial_k \partial_\ell \kappa_{\gamma mnij}) \left\{ -H_3^{mnk\ell} H_2^{ija} + 2H_3^{mnk\ell} (\partial_c H_3^{ijac}) \right\}, \tag{B3}
\end{aligned}$$

$$\begin{aligned}
\mathbb{H}_3^{ab} = & -2\kappa_{\gamma mnij} \left\{ H_1^{mn} H_3^{ijab} - H_2^{mna} H_2^{ijb} + 2H_2^{mna} (\partial_c H_3^{ijbc}) + H_2^{mn\ell} (\partial_\ell H_3^{ijab}) + H_3^{mnab} H_1^{ij} \right. \\
& - H_3^{mnab} (\partial_c H_2^{ijc}) - H_3^{mnal} (\partial_\ell H_2^{ijb}) - H_3^{mnka} (\partial_k H_2^{ijb}) + H_3^{mnab} (\partial_d \partial_c H_3^{ijdc}) \\
& + 2H_3^{mnal} (\partial_\ell \partial_c H_3^{ijbc}) + 2H_3^{mnka} (\partial_k \partial_c H_3^{ijbc}) + H_3^{mnk\ell} (\partial_k \partial_\ell H_3^{ijab}) \left. \right\} \\
& - 2(\partial_\ell \kappa_{\gamma mnij}) \left\{ H_2^{mn\ell} H_3^{ijab} - 2H_3^{mn\ell a} H_2^{ijb} + 4H_3^{mn\ell a} (\partial_c H_3^{ijbc}) + 2H_3^{mn\ell k} (\partial_k H_3^{ijab}) \right\} \\
& - 2(\partial_k \partial_\ell \kappa_{\gamma mnij}) H_3^{mnk\ell} H_3^{ijab}, \tag{B4}
\end{aligned}$$

$$\begin{aligned}
\mathbb{H}_4^{abc} = & -2\kappa_{\gamma mnij} \left\{ H_2^{mna} H_3^{ijbc} - H_3^{mnab} H_2^{ijc} + 2H_3^{mnab} (\partial_e H_3^{ijce}) + H_3^{mnal} (\partial_\ell H_3^{ijbc}) + H_3^{mnka} (\partial_k H_3^{ijbc}) \right\} \\
& - 4(\partial_k \kappa_{\gamma mnij}) H_3^{mnka} H_3^{ijbc}, \tag{B5}
\end{aligned}$$

$$\mathbb{H}_5^{abcd} = -2\kappa_{\gamma mnij} H_3^{mnab} H_3^{ijcd}, \tag{B6}$$

$$\begin{aligned}
\mathbb{H}_{6a} = & -2\alpha^{(3)} \Gamma_{ba}^b - 4\alpha_{,a} - \kappa_{\gamma mnij} \left\{ 2H_1^{mn} M_{1a}^{ij} - 2H_1^{mn} (\partial_d M_{2a}^{ijd}) - H_1^{mn} \mathcal{M}^{(i} \delta^{j)}_a + 2H_2^{mn\ell} (\partial_\ell M_{1a}^{ij}) \right. \\
& - 2H_2^{mn\ell} (\partial_\ell \partial_d M_{2a}^{ijd}) + 2H_3^{mnk\ell} (\partial_k \partial_\ell M_{1a}^{ij}) - 2H_3^{mnk\ell} (\partial_k \partial_\ell \partial_d M_{2a}^{ijd}) \left. \right\} \\
& - (\partial_\ell \kappa_{\gamma mnij}) \left\{ 2H_2^{mn\ell} M_{1a}^{ij} - 2H_2^{mn\ell} (\partial_d M_{2a}^{ijd}) - H_2^{mn\ell} \mathcal{M}^{(i} \delta^{j)}_a + 4H_3^{mn\ell k} (\partial_k M_{1a}^{ij}) \right. \\
& - 4H_3^{mn\ell k} (\partial_k \partial_d M_{2a}^{ijd}) \left. \right\} - (\partial_k \partial_\ell \kappa_{\gamma mnij}) \left\{ 2H_3^{mnk\ell} M_{1a}^{ij} - 2H_3^{mnk\ell} (\partial_d M_{2a}^{ijd}) - H_3^{mnk\ell} \mathcal{M}^{(i} \delta^{j)}_a \right\} \\
& - \kappa_{Kmnij} \left\{ 2H_4^{mn} M_{3a}^{ij} - 2H_4^{mn} (\partial_\ell M_{4a}^{ij\ell}) \right\}, \tag{B7}
\end{aligned}$$

$$\begin{aligned}
\mathbb{H}_{7a}{}^b = & -2\alpha\delta_a^b - \kappa_{\gamma mnij} \left\{ -2H_1^{mn}M_{2a}{}^{ijb} + 2H_2^{mnb}M_{1a}{}^{ij} - 2H_2^{mnb}(\partial_d M_{2a}{}^{ijd}) - 2H_2^{mn\ell}(\partial_\ell M_{2a}{}^{ijb}) \right. \\
& - H_2^{mnb}\mathcal{M}^j\delta_a^i - H_2^{mnb}\mathcal{M}^i\delta_a^j + 2H_3^{mnbl}(\partial_\ell M_{1a}{}^{ij}) + 2H_3^{mnkb}(\partial_k M_{1a}{}^{ij}) - 2H_3^{mnbl}(\partial_\ell\partial_d M_{2a}{}^{ijd}) \\
& - 2H_3^{mnkb}(\partial_k\partial_d M_{2a}{}^{ijd}) - 2H_3^{mnkl}(\partial_k\partial_\ell M_{2a}{}^{ijb}) - H_3^{mnbl}(\partial_\ell\mathcal{M}^{(i)}\delta^j)_a - H_3^{mnbl}(\partial_\ell\mathcal{M}^{(j)}\delta^i)_a \left. \right\} \\
& - (\partial_\ell\kappa_{\gamma mnij}) \left\{ -2H_2^{mn\ell}M_{2a}{}^{ijb} + 4H_3^{mn\ell b}M_{1a}{}^{ij} - 4H_3^{mn\ell b}(\partial_d M_{2a}{}^{ijd}) - 4H_3^{mn\ell k}(\partial_k M_{2a}{}^{ijb}) \right. \\
& - 2H_3^{mn\ell b}\mathcal{M}^j\delta_a^i - 2H_3^{mn\ell b}\mathcal{M}^i\delta_a^j \left. \right\} + 2(\partial_k\partial_\ell\kappa_{\gamma mnij})H_3^{mnkl}M_{2a}{}^{ijb} + 2\kappa_{Kmnij}H_4^{mn}M_{4a}{}^{ijb}, \quad (B8)
\end{aligned}$$

$$\begin{aligned}
\mathbb{H}_{8a}{}^{bc} = & -\kappa_{\gamma mnij} \left\{ -2H_2^{mnb}M_{2a}{}^{ijc} + 2H_3^{mnbc}M_{1a}{}^{ij} - 2H_3^{mnbc}(\partial_d M_{2a}{}^{ijd}) - 2H_3^{mnbl}(\partial_\ell M_{2a}{}^{ijc}) \right. \\
& - 2H_3^{mnkb}(\partial_k M_{2a}{}^{ijc}) - H_3^{mnbc}\mathcal{M}^j\delta_a^i - H_3^{mnbc}\mathcal{M}^i\delta_a^j \left. \right\} + 4(\partial_k\kappa_{\gamma mnij})H_3^{mnkb}M_{2a}{}^{ijc}, \quad (B9)
\end{aligned}$$

$$\mathbb{H}_{9a}{}^{bcd} = 2\kappa_{\gamma mnij}H_3^{mnbc}M_{2a}{}^{ijd}. \quad (B10)$$

The propagation equation of the momentum constraint with  $C^2$ -adjusted ADM formulation can be written as

$$\partial_t\mathcal{M}_a = \mathbb{M}_{1a}\mathcal{H} + \mathbb{M}_{2a}{}^b(\partial_b\mathcal{H}) + \mathbb{M}_{3a}{}^{bc}(\partial_b\partial_c\mathcal{H}) + \mathbb{M}_{4a}{}^{bcd}(\partial_b\partial_c\partial_d\mathcal{H}) + \mathbb{M}_{5ab}\mathcal{M}^b + \mathbb{M}_{6ab}{}^c(\partial_c\mathcal{M}^b) + \mathbb{M}_{7ab}{}^{cd}(\partial_c\partial_d\mathcal{M}^b), \quad (B11)$$

where

$$\begin{aligned}
\mathbb{M}_{1a} = & -\alpha_{,a} - 2\kappa_{\gamma mnij} \left\{ M_{1a}{}^{mn}H_1{}^{ij} - M_{1a}{}^{mn}(\partial_c H_2{}^{ijc}) + M_{1a}{}^{mn}(\partial_d\partial_c H_3{}^{ijdc}) + M_{2a}{}^{mn\ell}(\partial_\ell H_1{}^{ij}) \right. \\
& - M_{2a}{}^{mn\ell}(\partial_\ell\partial_c H_2{}^{ijc}) + M_{2a}{}^{mn\ell}(\partial_\ell\partial_d\partial_c H_3{}^{ijdc}) \left. \right\} - 2(\partial_\ell\kappa_{\gamma mnij}) \left\{ M_{2a}{}^{mn\ell}H_1{}^{ij} - M_{2a}{}^{mn\ell}(\partial_c H_2{}^{ijc}) \right. \\
& + M_{2a}{}^{mn\ell}(\partial_d\partial_c H_3{}^{ijdc}) \left. \right\} - 2\kappa_{Kmnij} \left\{ M_{3a}{}^{mn}H_4{}^{ij} + M_{4a}{}^{mn\ell}(\partial_\ell H_4{}^{ij}) \right\} - 2(\partial_\ell\kappa_{Kmnij})M_{4a}{}^{mn\ell}H_4{}^{ij}, \quad (B12)
\end{aligned}$$

$$\begin{aligned}
\mathbb{M}_{2a}{}^b = & -\frac{1}{2}\alpha\delta_a^b - 2\kappa_{\gamma mnij} \left\{ -M_{1a}{}^{mn}H_2{}^{ijb} + 2M_{1a}{}^{mn}(\partial_c H_3{}^{ijbc}) + M_{2a}{}^{mn\ell}H_1{}^{ij} - M_{2a}{}^{mn\ell}(\partial_c H_2{}^{ijc}) \right. \\
& - M_{2a}{}^{mn\ell}(\partial_\ell H_2{}^{ijb}) + M_{2a}{}^{mn\ell}(\partial_d\partial_c H_3{}^{ijdc}) + 2M_{2a}{}^{mn\ell}(\partial_\ell\partial_c H_3{}^{ijbc}) \left. \right\} \\
& - 2(\partial_\ell\kappa_{\gamma mnij}) \left\{ -M_{2a}{}^{mn\ell}H_2{}^{ijb} + 2M_{2a}{}^{mn\ell}(\partial_c H_3{}^{ijbc}) \right\} - 2\kappa_{Kmnij}M_{4a}{}^{mn\ell}H_4{}^{ij}, \quad (B13)
\end{aligned}$$

$$\begin{aligned}
\mathbb{M}_{3a}{}^{bc} = & -2\kappa_{\gamma mnij} \left\{ M_{1a}{}^{mn}H_3{}^{ijbc} - M_{2a}{}^{mn\ell}H_2{}^{ijc} + 2M_{2a}{}^{mn\ell}(\partial_d H_3{}^{ijcd}) + M_{2a}{}^{mn\ell}(\partial_\ell H_3{}^{ijbc}) \right\} \\
& - 2(\partial_\ell\kappa_{\gamma mnij})M_{2a}{}^{mn\ell}H_3{}^{ijbc}, \quad (B14)
\end{aligned}$$

$$\mathbb{M}_{4a}{}^{bcd} = -2\kappa_{\gamma mnij}M_{2a}{}^{mn\ell}H_3{}^{ijcd}, \quad (B15)$$

$$\begin{aligned}
\mathbb{M}_{5ab} = & \gamma_{mb}\beta^m{}_{,a} + \beta^\ell\gamma_{ab,\ell} + \alpha K\gamma_{ab} - \kappa_{\gamma mnij} \left\{ 2M_{1a}{}^{mn}M_{1b}{}^{ij} - 2M_{1a}{}^{mn}(\partial_d M_{2b}{}^{ijd}) - M_{1a}{}^{mn}\mathcal{M}^{(j}\delta^i)_{,b} \right. \\
& + 2M_{2a}{}^{mn\ell}(\partial_\ell M_{1b}{}^{ij}) - 2M_{2a}{}^{mn\ell}(\partial_\ell\partial_d M_{2b}{}^{ijd}) \left. \right\} - (\partial_\ell\kappa_{\gamma mnij}) \left\{ 2M_{2a}{}^{mn\ell}M_{1b}{}^{ij} - 2M_{2a}{}^{mn\ell}(\partial_d M_{2b}{}^{ijd}) \right. \\
& - M_{2a}{}^{mn\ell}\mathcal{M}^{(j}\delta^i)_{,b} \left. \right\} - 2\kappa_{Kmnij} \left\{ M_{3a}{}^{mn}M_{3b}{}^{ij} - M_{3a}{}^{mn}(\partial_\ell M_{4b}{}^{ij\ell}) + M_{4a}{}^{mn\ell}(\partial_\ell M_{3b}{}^{ij}) \right. \\
& - M_{4a}{}^{mn\ell}(\partial_\ell\partial_d M_{4b}{}^{ijd}) \left. \right\} - 2(\partial_\ell\kappa_{Kmnij}) \left\{ M_{4a}{}^{mn\ell}M_{3b}{}^{ij} - M_{4a}{}^{mn\ell}(\partial_d M_{4b}{}^{ijd}) \right\}, \quad (B16)
\end{aligned}$$

$$\begin{aligned}
\mathbb{M}_{6ab}{}^c = & \beta^c \gamma_{ab} - \kappa_{\gamma mnij} \left\{ -2M_{1a}{}^{mn} M_{2b}{}^{ijc} + 2M_{2a}{}^{mnc} M_{1b}{}^{ij} - 2M_{2a}{}^{mnc} (\partial_d M_{2b}{}^{ijd}) - 2M_{2a}{}^{mn\ell} (\partial_\ell M_{2b}{}^{ijc}) \right. \\
& \left. - M_{2a}{}^{mnc} \mathcal{M}^j \delta^i{}_b - M_{2a}{}^{mnc} \mathcal{M}^i \delta^j{}_b \right\} + 2(\partial_\ell \kappa_{\gamma mnij}) M_{2a}{}^{mn\ell} M_{2b}{}^{ijc} \\
& - 2\kappa_{Kmnij} \left\{ -M_{3a}{}^{mn} M_{4b}{}^{ijc} + M_{4a}{}^{mnc} M_{3b}{}^{ij} - M_{4a}{}^{mnc} (\partial_d M_{4b}{}^{ijd}) - M_{4a}{}^{mn\ell} (\partial_\ell M_{4b}{}^{ijc}) \right\} \\
& + 2(\partial_\ell \kappa_{Kmnij}) M_{4a}{}^{mn\ell} M_{4b}{}^{ijc}, \tag{B17}
\end{aligned}$$

$$\mathbb{M}_{7ab}{}^{cd} = 2\kappa_{\gamma mnij} M_{2a}{}^{mnc} M_{2b}{}^{ijd} + 2\kappa_{Kmnij} M_{4a}{}^{mnc} M_{4b}{}^{ijd}. \tag{B18}$$

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